A Criterion for the Viability of Stochastic Semilinear Control Systems via the Quasi-Tangency Condition

Dan Goreac

Université Paris-Est, Laboratoire d'Analyse et Mathématiques Appliquées, UMR 8050, 5, Boulevard Descartes, Cité Descartes - Champs-sur-Marne, 77454 Marne-la-Vallée, Cedex 2 e-mail: `Dan.Goreac@univ-mlv.fr`

Abstract   In this paper we study a criterion for the viability of stochastic semilinear control systems on a real, separable Hilbert space. The necessary and sufficient conditions are given using the notion of stochastic quasi-tangency. As a consequence, we prove that approximate viability and the viability property coincide for stochastic linear control systems. We obtain Nagumo's stochastic theorem and we present a method allowing to provide explicit criteria for the viability of smooth sets. We analyze the conditions characterizing the viability of the unit ball. The paper generalizes recent results from the deterministic framework.

Keywords: viability, stochastic control, semilinear control systems
AMS Classification: 60H15, 35R60, 93E20

1 Introduction

We begin by introducing the basic notations. We consider two separable real Hilbert spaces $(H, \langle \cdot, \cdot \rangle_H)$, $(\Xi, \langle \cdot, \cdot \rangle_\Xi)$. We let $\mathcal{L}(\Xi; H)$ be the space of continuous linear operators and $L_2(\Xi; H)$ be the space of Hilbert-Schmidt linear operators endowed with its usual norm. We consider a linear operator $A : D(A) \subset H \longrightarrow H$ which generates a C_0-semigroup of linear operators $(S(t))_{t \geq 0}$. We let $(\Omega, \mathcal{F}, P)$ be a complete probability space. The process W will denote a cylindrical Wiener process with values in Ξ. The probability space $(\Omega, \mathcal{F}, P)$ is endowed with the natural, complete filtration $(\mathcal{F}_t)_{t \geq 0}$ generated by W. We consider $(G, \langle \cdot, \cdot \rangle_G)$ a real separable Hilbert space and a closed, bounded subset $U \subset G$. For a finite time horizon $T > 0$, we let $\mathcal{A}$ denote the space of all predictable processes $u : [0, T] \times \Omega \longrightarrow U$. The coefficient functions $f : H \times U \longrightarrow H$ and $g : H \times U \longrightarrow \mathcal{L}(\Xi; H)$ will

be supposed to satisfy standard assumptions (see Section 2). Finally, we consider a closed set $K \subset H$.

Given a stochastic control system

$$\begin{cases} dX^{t,\xi,u}(s) = \left(AX^{t,\xi,u}(s) + f\left(X^{t,\xi,u}(s), u\left(s\right)\right) \right) ds \\ + g\left(X^{t,\xi,u}(s), u\left(s\right)\right) dW_s, \text{ for all } s \in [t,T], \\ X^{t,\xi,u}(t) = \xi \in L^2\left(\Omega, \mathcal{F}_t, P; H\right), \end{cases} \tag{1}$$

the aim of this paper is to give necessary and sufficient conditions for which, for every $t \in [0,T]$, and every $\xi \in L^2\left(\Omega, \mathcal{F}_t, P; K\right)$, one can find an admissible control process $u \in \mathcal{A}$ such that the mild solution of (1) associated to u remains inside the set K, or, at least, in an arbitrarily small neighborhood of K. These properties are called viability, respectively $\varepsilon-$viability, and they have been extensively studied both in deterministic and stochastic setting. In the finite-dimensional deterministic framework, the first result on viability goes back to Nagumo [16] and it has been rediscovered several times in the late sixties. For stochastic finite-dimensional systems, the methods used to characterize viability rely either on stochastic contingent cones (e.g. [1]-[3], [13]) or on viscosity solutions (e.g. [4]-[6], [14]). We also recall [7] for a necessary condition for the viability of semilinear evolution systems using viscosity solutions of a class of fully nonlinear Hamilton-Jacobi-Bellman equations in abstract Hilbert spaces.

In [15] it has been proved by duality methods that approximate controllability of infinite-dimensional linear systems reduces to the study of (backward) viability with respect to some dual system. Therefore, beside the interest in the result itself, a necessary and sufficient criterion for viability in the infinite-dimensional setting could be a tool for the study of controllability properties.

Recently, the authors of [8]-[10] have provided a characterization of the viability of (deterministic) multi-valued nonlinear evolutions on Banach spaces via the quasi-tangency condition. Motivated by this approach, our main objective is to introduce the notion of quasi-tangency corresponding to the stochastic framework and to prove the relation with the viability of stochastic semilinear control systems. This will allow to extend existing results (e.g. [2]) in both directions: to an infinite-dimensional state space and to systems driven by cylindrical Brownian motion. Under the assumption that the noise coefficient g is $L_2\left(\Xi; H\right)$-valued, we provide a criterion equivalent to the ε-viability property. In the more general framework, this condition remains sufficient for ε-viability. For the converse, the estimates only allow to prove a slightly weaker quasi-tangency condition.

We point out that the stochastic quasi-tangency conditions extend the stochastic contingent cone introduced in [2] for the finite-dimensional setting. As for [2], the criteria are not deterministic (as is the case in [4]-[6], [14]), but the method can be easily adapted for random sets of constraints. One can derive deterministic conditions by calculating contingent sets to direct images. We will give a simple example showing how the viability of the unit ball can be explicitly characterized from the quasi-tangency conditions.

The paper is organized as follows: In the first section, we introduce the concept of quasi-tangency and state the main results. The second section is concerned with the proof of the equivalence between stochastic quasi-tangency and the property of ε-viability for stochastic semilinear control systems. In the last section, two classes of examples are considered. First, we prove that, for infinite-dimensional stochastic linear control systems, ε-viability and viability coincide. Second, we give Nagumo's stochastic theorem as a corollary of our main result and deduce explicit conditions for the viability of smooth sets. In particular, we study the viability of the unit ball in H.

2 Assumptions and main results

The coefficient functions $f : H \times U \longrightarrow H$ and $g : H \times U \longrightarrow \mathcal{L}(\Xi; H)$ are supposed to satisfy the following standard assumptions:

(A1) There exists some positive constant $c > 0$ such that

$$\begin{cases} |f(x,u) - f(y,u)| \leq c|x - y|, \text{ and} \\ |f(x,u)| \leq c(1 + |x|), \end{cases} \tag{A1}$$

for all $x, y \in H$ and all $u \in U$;

(A2) For every $t > 0$ and every $(x, u) \in H \times U$, one has $S(t)g(x,u) \in L_2(\Xi; H)$. Moreover, for some constants $c > 0$ and $0 \leq \gamma < \frac{1}{2}$,

$$\begin{cases} |g(x,u)|_{\mathcal{L}(\Xi;H)} \leq c(1 + |x|), \\ |S(t)(g(x,u) - g(y,u))|_{L_2(\Xi;H)} \leq c(t \wedge 1)^{-\gamma}|x - y|, \text{ and} \\ |S(t)g(x,u)|_{L_2(\Xi;H)} \leq c(t \wedge 1)^{-\gamma}(1 + |x|), \end{cases} \tag{A2}$$

for all $x, y \in H$ and all $u \in U$.

Given $t \in [0, T]$ and an admissible control process $u \in \mathcal{A}$, we recall that an $(\mathcal{F}_t)$-predictable process $X^{t,\xi,u}$ with $E\left[\sup_{s \in [t,T]} \left|X^{t,\xi,u}(s)\right|^2\right] < \infty$ is a mild solution of (1) if, for all $s \in [t, T]$,

$$X^{t,\xi,u}(s) = S(s-t)\xi + \int_t^s S(s-r)f\left(X^{t,\xi,u}(r), u(r)\right)dr$$

$$+ \int_t^s S(s-r)g\left(X^{t,\xi,u}(r), u(r)\right)dW_r, \ dP - a.s.$$

Under the standard assumptions (A1) and (A2), there exists a unique mild solution of (1). For further results on mild solutions, the reader is referred to [11] and [12].

Let us consider $\lambda \in \left[0, \frac{1}{2}\right)$ and introduce the concept of stochastic λ quasi-tangency.

Definition 1 *(λ Quasi-tangency condition) A closed set $K \subset H$ satisfies the λ quasi-tangency condition with respect to the control system (1) if, for every $t \in [0, T)$ and every $\xi \in L^2(\Omega, \mathcal{F}_t, P; K)$, we have*

$$\liminf_{h \searrow 0} \inf \left\{ \frac{1}{h^{1-2\lambda}} E\left[|\zeta - \eta|^2 \right] + \frac{1}{h^2} E\left[\left| E^{\mathcal{F}_t}\left[\zeta - \eta \right] \right|^2 \right] : \right.$$
$$\left. (\zeta, \eta) \in \mathcal{S}(t, h)\xi \times L^2(\Omega, \mathcal{F}_{t+h}, P; K) \right\} = 0, \qquad (2)$$

where

$$\mathcal{S}(t, h)\xi = \left\{ S(h)\xi + \int_t^{t+h} S(t+h-s) f(\xi, u(s)) \, ds, \right.$$

$$\left. + \int_t^{t+h} S(t+h-s) g(\xi, u(s)) \, dW_s, \ u \in \mathcal{A} \right\}.$$

Remark 1 The term involving the conditional expectation in (2) corresponds to the deterministic quasi-tangency condition. The term in $h^{1-2\lambda}$ (written for $\lambda = \gamma$) is specific to the stochastic part of the equation (1). Whenever the coefficient function g takes its values in $L_2(\Xi; H)$ and is Lipschitz continuous in the state variable, one can consider $\lambda = 0$.

The following simple proposition provides a sequential formulation of the stochastic quasi-tangency condition.

Proposition 1 *A nonempty, closed set $K \subset H$ satisfies the λ quasi-tangency condition with respect to the control system (1) if and only if, for every $t \in [0, T)$ and every $\xi \in L^2(\Omega, \mathcal{F}_t, P; K)$, there exist a sequence of positive real constants $h_n \searrow 0$, a sequence of random variables $p_n \in L^2(\Omega, \mathcal{F}_{t+h_n}, P; H)$ and a sequence of admissible control processes $(u_n)_n \subset \mathcal{A}$ such that the following assertions hold simultaneously:*

(a) $\lim_n \left(E\left[|p_n|^2 \right] + \frac{1}{h_n^{1+2\gamma}} E\left[\left| E^{\mathcal{F}_t}\left[p_n \right] \right|^2 \right] \right) = 0$, *and*

(b) $S(h_n)\xi + \int_t^{t+h_n} S(t+h_n - s) f(\xi, u_n(s)) \, ds +$

$\int_t^{t+h_n} S(t+h_n - s) g(\xi, u_n(s)) \, dW_s + h_n^{\frac{1}{2}-\lambda} p_n \in K$, $dP-$*almost surely, for all $n \geq 1$.*

The proof is straightforward and we leave the details to the interested reader. We recall the definitions of $\varepsilon-$viability and viability.

Definition 2 *(a) A nonempty, closed set $K \subset H$ is called (mild) viable with respect to the control system (1) if, for every $t \in [0, T]$ and every initial condition $\xi \in L^2(\Omega, \mathcal{F}_t, P; K)$, there exists an admissible control process u such that*

$$X^{t,\xi,u}(s) \in K, \ dP\text{-a.s., for all } s \in [t, T].$$

(b) A nonempty, closed set $K \subset H$ is called (mild) $\varepsilon-$viable with respect to the control system (1) if, for every $t \in [0, T]$ and every initial condition $\xi \in L^2(\Omega, \mathcal{F}_t, P; K)$,

$$\inf_{u \in \mathcal{A}} \sup_{s \in [t, T]} d\left(X^{t,\xi,u}(s), L^2(\Omega, \mathcal{F}_s, P; K) \right) = 0.$$

Remark 2 In the definition of $\varepsilon-$viability, the set $L^2\left(\Omega, \mathcal{F}_s, P; K\right)$ should be seen as a closed subset of $L^2\left(\Omega, \mathcal{F}_s, P; H\right)$ and $d\left(\cdot, L^2\left(\Omega, \mathcal{F}_s, P; K\right)\right)$ as the usual distance to a closed set in the Hilbert space $L^2\left(\Omega, \mathcal{F}_s, P; H\right),$ i.e.

$$d^2\left(\zeta, L^2\left(\Omega, \mathcal{F}_s, P; K\right)\right) = \inf\left\{E\left[\left|\zeta - \eta\right|^2\right] : \eta \in L^2\left(\Omega, \mathcal{F}_s, P; K\right)\right\},$$

for every $\zeta \in L^2\left(\Omega, \mathcal{F}_s, P; H\right).$ Whenever the projection map $\Pi_K : H \rightsquigarrow K$ defined by

$$\Pi_K(x) = \left\{y \in K : d_K(x) = |y - x|\right\}$$

has nonempty images (e.g. for closed, convex sets of constraints), using [3], Corollary 8.2.13, one can replace the ε-viability condition by

$$\inf_{u \in \mathcal{A}} \sup_{s \in [t,T]} E\left[d_K^2\left(X^{t,\xi,u}(s)\right)\right] = 0.$$

We now state the main results of the paper. The proofs will be postponed to the next section.

Theorem 1 *(Necessary condition for ε-viability)*
Let us suppose that (A1) and (A2) hold true. Moreover, we suppose that $K \subset H$ is a nonempty, closed set. If K is $\varepsilon-$viable with respect to the control system (1), then the quasi-tangency condition (2) holds true for $\lambda = \gamma.$

Theorem 2 *(Sufficient condition for ε-viability)*
Let us suppose that (A1) and (A2) hold true. Moreover, we suppose that $K \subset H$ is a nonempty, closed set and the quasi-tangency condition (2) holds true for $\lambda = 0.$ Then the set K is $\varepsilon-$viable with respect to the control system (1).

The following result is a direct consequence of the previous theorems. It provides a useful criterion equivalent to the ε-viability of control systems with $L_2\left(\Xi; H\right)$-valued noise coefficient.

Corollary 1 *We assume (A1) and*
    (A2') For every $(x, u) \in H \times U$, one has $g(x, u) \in L_2\left(\Xi; H\right).$ Moreover, for some real constant $c > 0,$

$$\begin{cases} \left|g(x, u) - g(y, u)\right|_{L_2(\Xi; H)} \leq c\left|x - y\right|, \text{ and} \\ \left|g(x, u)\right|_{L_2(\Xi; H)} \leq c\left(1 + |x|\right), \end{cases} \tag{A2'}$$

for all $x, y \in H$ and all $u \in U.$
    Then the quasi-tangency condition (2) with $\lambda = 0$ provides a criterion equivalent to the ε-viability property.

3 Proof of the main results

3.1 Necessary condition for ε-viability

Proof (of Theorem 1). We begin by proving that, whenever K enjoys the ε-viability property, it satisfies the quasi-tangency condition with $\lambda = \gamma$. We consider arbitrary $t \in [0, T)$, $h \in (0, 1)$ small enough and $\xi \in L^2(\Omega, \mathcal{F}_t, P; K)$. If K enjoys the ε-viability property, then there exists an admissible control process u (which may depend on t, h and ξ) such that the mild solution of (1) issued from ξ and associated to u (denoted by $X^{t,\xi,u}$) satisfies

$$d\left(X^{t,\xi,u}(s), L^2(\Omega, \mathcal{F}_s, P; K)\right) < h^3 \text{ for all } s \in [t, T].$$

In particular, there exists a random variable $\eta \in L^2(\Omega, \mathcal{F}_{t+h}, P; K)$ such that

$$E\left[\left|X^{t,\xi,u}(t+h) - \eta\right|^2\right] < h^3. \tag{3}$$

Using the continuity property of the mild solution (see, for example [11] proof of Theorem 9.9.1), we get

$$\sup_{s \in [t,t+h]} E\left[\left|X^{t,\xi,u}(s) - \xi\right|^2\right] \leq C\left(\sup_{r \in [0,h]} E\left[\left|S(r)\xi - \xi\right|^2\right] + h^{1-2\gamma}\right), \tag{4}$$

where C is a generic constant which may change from one line to another. C depends on the Lipschitz constants of the coefficients, the initial data ξ and the time horizon $T > 0$ (but not on h). Using the assumptions (A1) and (A2) and the inequality (4), one gets

$$\sup_{s \in [t,t+h]} E\left[\left|f\left(X^{t,\xi,u}(s), u(s)\right) - f(\xi, u(s))\right|^2\right]$$
$$\leq C\left(\sup_{r \in [0,h]} E\left[\left|S(r)\xi - \xi\right|^2\right] + h^{1-2\gamma}\right), \tag{5}$$

and

$$E\left[\left|S(t+h-s)\left(g\left(X^{t,\xi,u}(s), u(s)\right) - g(\xi, u(s))\right)\right|^2\right]$$
$$\leq C(t+h-s)^{-2\gamma}\left(\sup_{r \in [0,h]} E\left[\left|S(r)\xi - \xi\right|^2\right] + h^{1-2\gamma}\right), \tag{6}$$

for all $s \in [t, t+h]$. Let us now introduce

$$q_h = \eta - S(h)\xi - \int_t^{t+h} S(t+h-s)f(\xi, u(s))\, ds$$
$$- \int_t^{t+h} S(t+h-s)g(\xi, u(s))\, dW_s. \tag{7}$$

Combining (5), (6) and (3) yields

$$
\begin{aligned}
E&\left[|q_h|^2\right]\\
&\leq C\left(E\left[\left|X^{t,\xi,u}\left(t+h\right)-\eta\right|^2\right]\right.\\
&\quad+E\left[\left|\int_t^{t+h}S\left(t+h-s\right)\left[f\left(X^{t,\xi,u}(s),u(s)\right)-f\left(\xi,u(s)\right)\right]ds\right|^2\right]\\
&\quad\left.+E\left[\left|\int_t^{t+h}S\left(t+h-s\right)\left[g\left(X^{t,\xi,u}(s),u(s)\right)-g\left(\xi,u(s)\right)\right]dW_s\right|^2\right]\right)\\
&\leq Ch^{1-2\gamma}\left(\sup_{r\in[0,h]}E\left[\left|S\left(r\right)\xi-\xi\right|^2\right]+h^{1-2\gamma}\right).
\end{aligned}
\tag{8}
$$

Next, we notice that

$$
E^{\mathcal{F}_t}\left[q_h\right]=E^{\mathcal{F}_t}\left[\eta\right]-S\left(h\right)\xi-\int_t^{t+h}S\left(t+h_n-s\right)f\left(\xi,u(s)\right)ds.
$$

Thus, using Jensen's inequality and (5), we get

$$
E\left[\left|E^{\mathcal{F}_t}\left[q_h\right]\right|^2\right]\leq Ch^2\left(h^{1-2\gamma}+\sup_{r\in[0,h]}E\left[\left|S\left(r\right)\xi-\xi\right|^2\right]\right).
\tag{9}
$$

We introduce the $\mathcal{F}_{t+h}$-measurable random variable

$$
p_h=h^{\gamma-\frac{1}{2}}q_h.
\tag{10}
$$

The inequalities (8) and (9) imply

$$
E\left[|p_h|^2\right]+\frac{1}{h^{1+2\gamma}}E\left[\left|E^{\mathcal{F}_t}\left[p_h\right]\right|^2\right]\leq C\left(\sup_{r\in[0,h]}E\left[\left|S\left(r\right)\xi-\xi\right|^2\right]+h^{1-2\gamma}\right).
\tag{11}
$$

Using the strong continuity of $(S(r))_{r\geq0}$ and a dominated convergence argument, we get

$$
\lim_{h\to0+}\left(E\left[|p_h|^2\right]+\frac{1}{h^{1+2\gamma}}E\left[\left|E^{\mathcal{F}_t}\left[p_h\right]\right|^2\right]\right)=0.
$$

Also, by the choice of η and p_h,

$$
\begin{aligned}
S\left(h\right)\xi&+\int_t^{t+h}S\left(t+h-s\right)f\left(\xi,u(s)\right)ds\\
&+\int_t^{t+h}S\left(t+h-s\right)g\left(\xi,u(s)\right)dW_s+h^{\left(\frac{1}{2}-\gamma\right)}p_h=\eta\in K,
\end{aligned}
$$

$dP-$almost surely. The proof is now complete.

Remark 3 1. A careful look at the previous proof shows that lim inf in the definition of quasi-tangency can be strengthen to lim sup .

2. Moreover, for every deterministic initial data $\xi = x \in K$ and every $k \geq 2$, the random variable p_h (given by (10)) can be chosen in $L^k (\Omega, \mathcal{F}_{t+h}, P; H)$ whenever one of the following assumptions holds true:

a) the set of constraints K is viable with respect to (1);

b) the set of constraints K is convex and ε-viable;

c) the set K is bounded and ε-viable.

Indeed, if a) holds true, then the random variable η in (3) can be chosen as $\eta = X_{t+h}^{t,x,u}$. Under the assumption b), one can choose $\eta = \pi_K \left(X_{t+h}^{t,x,u} \right)$, where π_K is the projection on K. If c) holds true, $\eta \in L^\infty (\Omega, \mathcal{F}_{t+h}, P; H)$. The conclusion follows from (7) and (10).

3.2 Sufficient condition for the ε-viability property

In order to prove the converse, we introduce the notion of ε-approximate mild solution.

Definition 3 *For every $0 \leq t \leq \widetilde{T} \leq T$, every initial condition $\xi \in L^2 (\Omega, \mathcal{F}_t, P; K)$ and every positive real constant ε, an ε-approximate mild solution of (1) defined on $\left[t, \widetilde{T} \right]$ is a sixtuple $(\sigma, u, \varphi, \psi, \theta, Y)$ such that*

(a) the function $\sigma : \left[t, \widetilde{T} \right] \longrightarrow \left[t, \widetilde{T} \right]$ is non decreasing and such that

$$ s - \varepsilon \leq \sigma (s) \leq s, \ \text{for all } s \in \left[t, \widetilde{T} \right]. $$

(b) the process u is an admissible control process.

(c) the process $\varphi : \left[t, \widetilde{T} \right] \longrightarrow H$ is predictable and such that

$$ E \left[\int_t^{\widetilde{T}} |\varphi (s)|^2 \, ds \right] \leq \left(\widetilde{T} - t \right) \varepsilon. $$

(d) the process $\psi : \left[t, \widetilde{T} \right] \longrightarrow L_2 (\Xi; H)$ is predictable and such that

$$ E \left[\int_t^{\widetilde{T}} |\psi (s)|_{L_2(\Xi;H)}^2 \, ds \right] \leq \left(\widetilde{T} - t \right) \varepsilon. $$

(e) the function $\theta : \left\{ (s,r) : t \leq r < s \leq \widetilde{T} \right\} \longrightarrow \left[t, \widetilde{T} \right]$ is Lebesgue-measurable and it satisfies $\theta (s,r) \leq s - t$ and $s \mapsto \theta (s,r)$ is nonexpansive on $\left(r, \widetilde{T} \right]$ (i.e. $|\theta (s,r) - \theta (s',r)| \leq |s - s'|$, for every $s, s' \in \left(r, \widetilde{T} \right]$).

(f) the process $Y : \left[t, \widetilde{T} \right] \longrightarrow H$ is predictable. Moreover, the process $Y_\sigma : \left[t, \widetilde{T} \right] \longrightarrow H$ defined by

$$ Y_\sigma (s) = Y (\sigma (s)), \ \text{for all } s \in \left[t, \widetilde{T} \right] $$

is predictable and

$$Y(s) = S(s-t)\xi + \int_t^s S(s-r) f(Y(\sigma(r)), u(r)) dr$$

$$+ \int_t^s S(s-r) g(Y(\sigma(r)), u(r)) dW_r$$

$$+ \int_t^s S(\theta(s,r)) \varphi(r) dr + \int_t^s S(\theta(s,r)) \psi(r) dW_r,$$

for all $s \in \left[t, \widetilde{T}\right]$.

(g) for every $s \in \left[t, \widetilde{T}\right]$, $Y(\sigma(s)) \in K$, dP-almost surely and $Y\left(\widetilde{T}\right) \in K$, dP-almost surely. Moreover,

$$E\left[|Y(\sigma(s)) - Y(s)|^2\right] \le \varepsilon, \text{ for all } s \in \left[t, \widetilde{T}\right].$$

We begin by proving some qualitative properties of ε-approximate mild solutions.

Proposition 2 *We suppose that (A1) and (A2) hold true. If $t, \widetilde{T} \in [0, T]$, such that $t \le \widetilde{T}$, $\xi \in L^2(\Omega, \mathcal{F}_t, P; K)$, $\varepsilon \in (0, 1)$ is a positive real constant and $(\sigma, u, \varphi, \psi, \theta, Y)$ is an ε-approximate mild solution of (1) defined on $\left[t, \widetilde{T}\right]$, then*

$$\sup_{s \in [t, \widetilde{T}]} E\left[|Y(s)|^2\right] \le C. \tag{12}$$

Here C is a positive real constant which only depends on T and ξ (but not on $t, \widetilde{T}$, ε nor on $(\sigma, u, \varphi, \psi, \theta, Y)$).

Proof Let us fix $s \in \left[t, \widetilde{T}\right]$. In order to prove (12), one uses the conditions (c) and (d) in Definition 3 to have

$$E\left[|Y(s)|^2\right] \le C\left(E\left[|S(s-t)\xi|^2\right]\right.$$

$$+ E\left[\left(\int_t^s |S(s-r) f(Y(\sigma(r)), u(r))| dr\right)^2\right]$$

$$+ E\left[\int_t^s |S(s-r) g(Y(\sigma(r), u(r)))|^2 dr\right] + \varepsilon\right)$$

$$= C(I_1 + I_2 + I_3 + \varepsilon). \tag{13}$$

To estimate I_1, we use the properties of the semigroup $(S(r))_{0 \le r \le T}$ and obtain

$$I_1 \le CE\left[|\xi|^2\right]. \tag{14}$$

For I_2, we write

$$
\begin{aligned}
I_2 &\leq CE\left[\left(\int_t^s |f\left(Y\left(\sigma\left(r\right)\right), u\left(r\right)\right)| \, dr\right)^2\right] \\
&\leq C\left(s-t\right)^{1+2\gamma} \int_t^s \left(s-r\right)^{-2\gamma} E\left[\left|f\left(Y\left(r\right), u\left(r\right)\right)\right|^2\right] dr \\
&\quad + C\int_t^s E\left[\left|Y\left(r\right)-Y\left(\sigma\left(r\right)\right)\right|^2\right] dr,
\end{aligned}
\tag{15}
$$

Using property (g) in Definition 3 and the assumption (A1), the inequality (15) yields

$$
I_2 \leq C\left(\int_t^s \left(s-r\right)^{-2\gamma} E\left[\left|Y\left(r\right)\right|^2\right] dr + 1\right).
\tag{16}
$$

Similar arguments allow to obtain

$$
I_3 \leq C\left(\int_t^s \left(\left(s-r\right)\wedge 1\right)^{-2\gamma} E\left[\left|Y\left(r\right)\right|^2\right] dr + 1\right).
\tag{17}
$$

We substitute (14), (16) and (17) in (13) to finally get

$$
E\left[\left|Y\left(s\right)\right|^2\right] \leq C\left(1 + \int_t^s \left(s-r\right)^{-2\gamma} E\left[\left|Y\left(r\right)\right|^2\right] dr\right),
$$

for all $s \in \left[t, \widetilde{T}\right]$. The conclusion follows from a variant of Gronwall's inequality.

The following result proves further regularity properties of the Y component of an approximate mild solution.

Proposition 3 *If $t, \widetilde{T} \in [0, T]$, such that $t \leq \widetilde{T}$, the initial condition $\xi \in L^2\left(\Omega, \mathcal{F}_t, P; K\right)$, $\varepsilon \in (0, 1)$ is a positive real constant and $(\sigma, u, \varphi, \psi, \theta, Y)$ is an ε-approximate mild solution of (1), then Y is mean-square continuous.*

Proof We let $(\sigma, u, \varphi, \psi, \theta, Y)$ be an ε-approximate mild solution of (1) defined on $\left[t, \widetilde{T}\right]$. Let us fix $s \in \left[t, \widetilde{T}\right]$. For every $s \leq s'$,

$$E\left[\left|Y\left(s'\right) - Y\left(s\right)\right|^2\right]$$

$$\leq C\left(E\left[\left|S(s'-s)\xi - \xi\right|^2\right]\right.$$

$$+ E\left[\left|\int_s^{s'} S\left(s'-r\right) f\left(Y\left(\sigma\left(r\right)\right), u\left(r\right)\right) dr\right|^2\right]$$

$$+ E\left[\left|\int_t^s \left(S\left(s'-r\right) - S\left(s-r\right)\right) f\left(Y\left(\sigma\left(r\right)\right), u\left(r\right)\right) dr\right|^2\right]$$

$$+ E\left[\left|\int_s^{s'} S\left(s'-r\right) g\left(Y\left(\sigma\left(r\right)\right), u\left(r\right)\right) dW_r\right|^2\right]$$

$$+ E\left[\left|\int_t^s \left(S\left(s'-r\right) - S\left(s-r\right)\right) g\left(Y\left(\sigma\left(r\right)\right), u\left(r\right)\right) dW_r\right|^2\right]$$

$$+ E\left[\left|\int_s^{s'} S\left(\theta\left(s',r\right)\right) \varphi(r) dr\right|^2 + \left|\int_s^{s'} S\left(\theta\left(s',r\right)\right) \psi\left(r\right) dW_r\right|^2\right]$$

$$+ E\left[\left|\int_t^s \left(S\left(\theta\left(s',r\right)\right) - S\left(\theta\left(s,r\right)\right)\right) \varphi(r) dr\right|^2\right]$$

$$\left. + E\left[\left|\int_t^s \left(S\left(\theta\left(s',r\right)\right) - S\left(\theta\left(s,r\right)\right)\right) \psi\left(r\right) dW_r\right|^2\right]\right)$$

$$= C\left(I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8\right) \qquad (18)$$

The strong continuity of the semigroup S and a simple dominated convergence argument yield

$$\lim_{s' \searrow s} I_1 = 0. \qquad (19)$$

For the term I_2 (respectively I_4) we use (A1) (respectively (A2)) and Proposition 2 to get

$$\begin{cases} I_2 \leq C\left(s'-s\right)^2 & \text{and} \\ I_4 \leq C\left(s'-s\right)^{1-2\gamma}. \end{cases} \qquad (20)$$

For the term I_3 (respectively I_7) we use the continuity of the semigroup S and a dominated convergence argument to have

$$\lim_{s' \searrow s} I_3 = 0 = \lim_{s' \searrow s} I_7. \qquad (21)$$

Next, one notices that

$$I_5 = E\left[\int_t^s \left|\left(S\left(s'-s\right) - I\right) S\left(s-r\right) g\left(Y\left(\sigma\left(r\right)\right), u\left(r\right)\right)\right|^2 dr\right]$$

and, from the dominated convergence theorem it follows that

$$\lim_{s' \searrow s} I_5 = 0. \tag{22}$$

Similar arguments hold true for I_8. Finally, the conditions (c) and (d) in Definition 3 imply

$$\lim_{s' \searrow s} I_6 = 0. \tag{23}$$

Combining (18)-(23), we prove the mean-square right-continuity of Y. Similar arguments give the left-continuity. The proof of the Proposition is now complete.

The main step in the proof of Theorem 2 consists in the construction of approximate mild solutions. To this purpose, we prove

Lemma 1 *Let us suppose that (A1) and (A2) hold true and that $K \subset H$ is a nonempty, closed set which satisfies the quasi-tangency condition (2) with $\lambda = 0$. Then, for every $t \in [0, T)$, every initial condition $\xi \in L^2(\Omega, \mathcal{F}_t, P; K)$, every time horizon $\widetilde{T} \in [t, T]$ and for each $\varepsilon \in (0, 1)$, there exists an ε-approximate mild solution of (1) denoted by $(\sigma, u, \varphi, \psi, \theta, Y)$ and defined on $\left[t, \widetilde{T}\right]$.*

Proof Let us fix $t \in [0, T)$, $\xi \in L^2(\Omega, \mathcal{F}_t, P; K)$, $\widetilde{T} \in [t, T]$ and $\varepsilon \in (0, 1)$. The proof of the Lemma will be given in three steps.

<u>Step 1.</u> We will first show the existence of an ε-approximate mild solution on some small interval $[t, t + \delta]$. We fix $\varepsilon' \in (0, \varepsilon)$. We will latter specify how ε' should be chosen. Using the quasi-tangency property of K, one gets the existence of some $\delta \in (0, \varepsilon')$, of an admissible control process u and of a random variable $p \in L^2(\Omega, \mathcal{F}_{t+\delta}, P; H)$ such that

$$E\left[|p|^2\right] + \frac{1}{\delta} E\left[\left|E^{\mathcal{F}_t}[p]\right|^2\right] \le \varepsilon'$$

and

$$S(\delta)\xi + \int_t^{t+\delta} S(t+\delta-s) f(\xi, u(s)) ds \tag{24}$$

$$+ \int_t^{t+\delta} S(t+\delta-s) g(\xi, u(s)) dW_s + \sqrt{\delta}p \in K, dP - \text{almost surely}.$$

We define the functions $\sigma : [t, t + \delta] \longrightarrow [t, t + \delta]$ by setting $\sigma(s) = t$, for all $s \in [t, t + \delta]$. Using the martingale representation theorem for the random variable p, we get the existence of some $L_2(\Xi; H)$-valued predictable process η, defined on $[t, t + \delta]$ such that

$$p = E^{\mathcal{F}_t}[p] + \int_t^{t+\delta} \eta_s dW_s, \ dP - a.s.$$

We introduce $\varphi : [t, t + \delta] \longrightarrow H$ given by

$$\varphi(s) = \frac{1}{\sqrt{\delta}} E^{\mathcal{F}_t} [p], \text{ for all } s \in [t, t + \delta],$$

and $\psi : [t, t + \delta] \longrightarrow L_2(\Xi; H)$ given by

$$\psi(s) = \sqrt{\delta} \eta_s, \text{ for all } s \in [t, t + \delta].$$

Moreover, we let $\theta : \{t \leq r < s \leq t + \delta\} \longrightarrow [t, t + \delta]$ be defined by

$$\theta(s, r) = 0, \text{ for all } t \leq r < s \leq t + \delta.$$

Next, we define a process Y by

$$Y(s) = S(s - t)\xi + \int_t^s S(s - r) f(\xi, u(r)) dr$$

$$+ \int_t^s S(s - r) g(\xi, u(r)) dW_r + \int_t^s \varphi(r) dr + \int_t^s \psi(r) dW_r,$$

for all $s \in [t, t + \delta]$. We claim that $(\sigma, u, \varphi, \psi, \theta, Y)$ is an ε-approximate mild solution. The conditions (a), (b), (e) and (f) of Definition 3 are obviously satisfied. From the choice of p, one gets

$$E\left[|p|^2\right] + \frac{1}{\delta} E\left[\left|E^{\mathcal{F}_t}[p]\right|^2\right]$$

$$= E\left[\left|E^{\mathcal{F}_t}[p]\right|^2\right] + E\left[\int_t^{t+\delta} |\eta_s|^2 ds\right] + \frac{1}{\delta} E\left[\left|E^{\mathcal{F}_t}[p]\right|^2\right] \leq \varepsilon'.$$

Thus, the conditions (c) and (d) are also satisfied. Hence, we only need to check the last condition of Definition 3. To this purpose, we recall that $\delta < \varepsilon'$ and write

$$E\left[|Y(s) - \xi|^2\right] \leq C \left(E\left[|S(s - t)\xi - \xi|^2\right] \right.$$

$$+ E\left[\left(\int_t^s |S(s - r) f(\xi, u(r))| dr\right)^2\right]$$

$$+ E\left[\int_t^s |S(s - r) g(\xi, u(r))|^2 dr\right]$$

$$+ E\left[\left(\int_t^s |\varphi(r)| dr\right)^2\right] + E\left[\int_t^s |\psi(r)|^2 dr\right] \right)$$

$$\leq C \left(\sup_{r \in [0, \varepsilon']} E\left[|S(r)\xi - \xi|^2\right] + I_1 + I_2 + \varepsilon' \right), \qquad (25)$$

for all $s \in [t, t + \delta]$. Using (A2), we notice that

$$I_2 \leq C(\varepsilon')^{1 - 2\gamma} \left(E\left[|\xi|^2\right] + 1 \right).$$

Similar estimates hold true for I_1. We return to (25) and get

$$E\left[|Y(s) - \xi|^2\right] \le C\left(\sup_{r\in[0,\varepsilon']} E\left[|S(r)\xi - \xi|^2\right] + (\varepsilon')^{1-2\gamma}\right),$$

for all $s \in [t, t + \delta]$. The constant C may be chosen to depend only on T, ξ and the Lipschitz constant of f and g (but not on δ, nor ε'). Thus, by considering small enough $\varepsilon' \in (0, \varepsilon)$, we may assume, without loss of generality, that

$$E\left[|Y(s) - \xi|^2\right] \le \varepsilon, \text{ for all } s \in [t, t + \delta].$$

By the choice of σ, $Y(\sigma(s)) = \xi \in K$, dP-almost surely, for all $s \in [t, t + \delta]$ and (24) implies that $Y(t + \delta) \in K$, dP-almost surely. It follows that the condition (g) is also satisfied and, thus, $(\sigma, u, \varphi, \psi, \theta, Y)$ is an ε-approximate mild solution.

   Step 2. To prove the existence of some approximate solution on the whole interval $\left[t, \widetilde{T}\right]$, we use the following result, also known as the Brezis-Browder Theorem:

Theorem 3 *Let $\mathcal{S}$ be a nonempty set, $\preceq \subset \mathcal{S} \times \mathcal{S}$ a preorder on $\mathcal{S}$ and let $\mathcal{N} : \mathcal{S} \longrightarrow \mathbb{R} \cup \{+\infty\}$ be an increasing function. Suppose that each increasing sequence in $\mathcal{S}$ is bounded from above. Then, for each $a_0 \in \mathcal{S}$, there exists an $\mathcal{N}$-maximal element $a^* \in \mathcal{S}$ such that $a_0 \preceq a^*$.*

   For proof of this result and further remarks, the reader is referred to Theorem 2.1.1 in [10] and references therein.

   We now return to the proof of the Lemma. We introduce the set $\mathcal{S}$ of all ε-approximate mild solutions defined on intervals of the form $[t, t + \alpha] \subset \left[t, \widetilde{T}\right]$. On this set, we define the following preorder relation $\preceq$: given two ε-approximate mild solutions $(\sigma_1, u_1, \varphi_1, \psi_1, \theta_1, Y_1)$ defined on $[t, t + \alpha_1]$, respectively $(\sigma_2, u_2, \varphi_2, \psi_2, \theta_2, Y_2)$ defined on $[t, t + \alpha_2]$, we write

$$(\sigma_1, u_1, \varphi_1, \psi_1, \theta_1, Y_1) \preceq (\sigma_2, u_2, \varphi_2, \psi_2, \theta_2, Y_2)$$

if $\alpha_1 \le \alpha_2$, $u_1 = u_2$, $\varphi_1 = \varphi_2$, $\psi_1 = \psi_2$ on $[t, t + \alpha_1] \times \Omega$ up to an evanescent set and $\theta_1 = \theta_2$ almost everywhere on $\{t \le r < s \le t + \alpha_1\}$. We consider an increasing arbitrary sequence in $\mathcal{S}$

$$\mathcal{L} = \{(\sigma_n, u_n, \varphi_n, \psi_n, \theta_n, Y_n) \text{ defined respectively on } [t, t + \alpha_n], \ n \in \mathbb{N}\}.$$

We define

$$\alpha = \sup_n \alpha_n.$$

If $\alpha = \alpha_n$ for some index, then the element $(\sigma_n, u_n, \varphi_n, \psi_n, Y_n)$ is an upper bound for $\mathcal{L}$. Otherwise, since σ_n are increasing functions satisfying (a), there exists the limit

$$\lim_{n\to\infty} \uparrow \sigma_n(\alpha_n) \in [t, t + \alpha].$$

This allows to define an increasing function $\sigma : [t, t+\alpha] \to [t, t+\alpha]$ by setting

$$\sigma(s) = \begin{cases} \sigma_n(s), & \text{if } s \in [t, t+\alpha_n], \\ \lim_{n\to\infty} \uparrow \sigma_n(t+\alpha_n), & \text{if } s = t+\alpha. \end{cases}$$

The function σ satisfies the condition (a) of Definition 3. We consider an element $u^0 \in U$ and introduce the control process

$$u(s) = 1_{[t, t+\alpha_n]}(s)u_n(s) + 1_{\{t+\alpha\}}(s)\,u^0,$$

for $s \in [t, t+\alpha]$. Next, we define

$$\varphi(s) = \begin{cases} \varphi_n(s), & \text{if } s \in [t, t+\alpha_n], \\ 0, & \text{if } s = t+\alpha. \end{cases}$$

and

$$\psi(s) = \begin{cases} \psi_n(s), & \text{if } s \in [t, t+\alpha_n], \\ 0, & \text{if } s = t+\alpha. \end{cases}$$

For every n, one can extend φ_n on $[t, t+\alpha]$ by setting

$$\varphi_n(s) = 0, \text{ for all } s \in (t+\alpha_n, t+\alpha].$$

Then φ is the pointwise limit of φ_n (except for an evanescent set). Thus, φ is predictable. Property (c) of ε-approximate mild solutions yields

$$E\left[\int_t^{t+\alpha} |\varphi_n(s)|^2\, ds\right] \le \alpha_n \varepsilon, \text{ for all } n \in \mathbb{N}.$$

Then, by Fatou's lemma, one gets

$$E\left[\int_t^{t+\alpha} |\varphi(s)|^2\, ds\right] \le \alpha\varepsilon, \tag{26}$$

and the condition (c) holds for φ. Moreover, a simple dominated convergence argument proves that $\varphi_n \to_n \varphi$ in $L^2(\Omega \times [t, t+\alpha]; H)$. The condition (d) follows in the same way. Next, one notices that whenever $m, k \in \mathbb{N}$ such that $m \ge k$, for every $r \in [t, t+\alpha_k)$, the nonexpansive property of θ_m and θ_k yields

$$|\theta_m(t+\alpha_m, r) - \theta_k(t+\alpha_k, r)| \le |\alpha_m - \alpha_k|.$$

This implies the existence of $\lim_n \theta_n(t+\alpha_n, r)$. We define

$$\theta(s, r) = \begin{cases} \theta_n(s, r), & \text{if } t \le r < s \le t+\alpha_n, \\ \lim_n \theta_n(t+\alpha_n, r), & \text{if } t \le r < s = t+\alpha. \end{cases}$$

We recall that

$$Y_n(t + \alpha_n)$$

$$= S\left(\alpha_n\right)\xi + \int_t^{t+\alpha} 1_{[t,t+\alpha_n]}\left(r\right) S\left(t + \alpha_n - r\right) f\left(Y_n\left(\sigma_n\left(r\right)\right), u_n\left(r\right)\right) dr$$

$$+ \int_t^{t+\alpha} 1_{[t,t+\alpha_n]}\left(r\right) S\left(t + \alpha_n - r\right) g\left(Y_n\left(\sigma_n\left(r\right)\right), u_n\left(r\right)\right) dW_r$$

$$+ \int_t^{t+\alpha} 1_{[t,t+\alpha_n]}\left(r\right) S\left(\theta\left(t + \alpha_n, r\right)\right) \varphi_n\left(r\right) dr$$

$$+ \int_t^{t+\alpha} 1_{[t,t+\alpha_n]}\left(r\right) S\left(\theta\left(t + \alpha_n, r\right)\right) \psi_n\left(r\right) dW_r.$$

Proposition 2 and a simple dominated convergence argument allow to obtain the existence of the limit

$$\lim_n Y_n(t + \alpha_n) \text{ in } L^2\left(\Omega, \mathcal{F}_{t+\alpha}, P; H\right).$$

Moreover, since K is closed, the limit is in K dP-almost surely. We can now define

$$Y\left(s\right) = \begin{cases} Y_n\left(s\right), & \text{if } s \in [t, t + \alpha_n], \\ \lim_n Y_n(t + \alpha_n), & \text{if } s = t + \alpha. \end{cases}$$

We notice that, whenever $\sigma\left(t + \alpha\right) > \sigma\left(t + \alpha_n\right)$, for all $n \geq 1$, by the mean-square continuity of Y, the process Y_σ can be seen as the pointwise limit of the sequence

$$Y_n\left(\sigma_n\left(\cdot\right)\right) 1_{[t,t+\alpha_n]}\left(\cdot\right) + Y\left(\sigma_n\left(t + \alpha_n\right)\right) 1_{(t+\alpha_n,t+\alpha]}\left(\cdot\right).$$

Thus, Y_σ is predictable.

Let us check the condition (g) of Definition 3. We need to show that $Y\left(\sigma\left(s\right)\right) \in K$, $dP - a.s.$ and for all $s \in [t, t + \alpha]$. If $s \leq t + \alpha_n$ for some n, then $Y\left(\sigma\left(s\right)\right) \in K$, $dP - a.s.$ Otherwise, using the fact that $\sigma\left(t + \alpha\right) = \lim_n \sigma_n\left(t + \alpha_n\right)$, and Y is mean-square continuous, we get $Y\left(\sigma\left(t + \alpha\right)\right) \in K$, $dP - a.s.$ In order to prove that $(\sigma, u, \varphi, \psi, \theta, Y)$ is an ε-approximate mild solution on $[t, t + \alpha]$ one only needs to verify

$$E\left[\left|Y\left(s\right) - Y\left(\sigma\left(s\right)\right)\right|^2\right] \leq \varepsilon,$$

for all $s \in [t, t + \alpha]$. If $s \leq t + \alpha_n$ for some n, we have nothing to prove. We recall that

$$E\left[\left|Y\left(t + \alpha_n\right) - Y\left(\sigma\left(t + \alpha_n\right)\right)\right|^2\right] \leq \varepsilon,$$

for all $n \geq 1$. Using the definition of Y and σ and the continuity of Y, we also get

$$E\left[\left|Y\left(t + \alpha\right) - Y\left(\sigma\left(t + \alpha\right)\right)\right|^2\right] \leq \varepsilon.$$

It follows that $(\sigma, u, \varphi, \psi, \theta, Y)$ is an ε-approximate mild solution on $[t, t + \alpha]$ and an upper bound for $\mathcal{L}$. We introduce the increasing function

$$\mathcal{N} : \mathcal{S} \longrightarrow \mathbb{R}_+, \text{ given by } \mathcal{N}\left((\sigma, u, \varphi, \psi, \theta, Y)\right) = \alpha,$$

whenever $(\sigma, u, \varphi, \psi, \theta, Y)$ is defined on $[t, t + \alpha]$. We apply the Brezis-Browder Theorem to obtain the existence of an $\mathcal{N}$-maximal element of $\mathcal{S}$ denoted by $(\sigma^*, u^*, \varphi^*, \psi^*, Y^*)$ and defined on $[t, t + \alpha^*]$.

    Step 3. We claim that $t + \alpha^* = \widetilde{T}$. Let us assume, for the moment, that $t + \alpha^* < \widetilde{T}$. By definition, $Y^*(t + \alpha^*) \in K$ dP-a.s. We recall that K satisfies the quasi-tangency condition with respect to the control system (1). Therefore, for every $\varepsilon' < \varepsilon$ there exist $0 < \delta^* \leq \min\left\{\widetilde{T} - t - \alpha^*, \varepsilon'\right\}$, $p^* \in L^2\left(\Omega, \mathcal{F}_{t+\alpha^*+\delta^*}, P; H\right)$ and an admissible control process $\widetilde{u} \in \mathcal{A}$ such that

$$E\left[|p^*|^2\right] + \frac{1}{\delta^*} E\left[\left|E^{\mathcal{F}_{t+\alpha^*}}\left[p^*\right]\right|^2\right] \leq \varepsilon', \text{ and} \tag{27}$$

$$S\left(\delta^*\right) Y^*\left(t + \alpha^*\right) + \int_{t+\alpha^*}^{t+\alpha^*+\delta^*} S\left(t + \alpha^* + \delta^* - s\right) f\left(Y^*\left(t + \alpha^*\right), \widetilde{u}(s)\right) ds$$

$$+ \int_{t+\alpha^*}^{t+\alpha^*+\delta^*} S\left(t + \alpha^* + \delta^* - s\right) g\left(Y^*\left(t + \alpha^*\right), \widetilde{u}(s)\right) dW_s + \sqrt{\delta^*} p^* \in K, \tag{28}$$

$dP-$almost surely. The martingale representation theorem yields the existence of some predictable process η^* defined on $[t + \alpha^*, t + \alpha^* + \delta^*]$ such that

$$p^* = E^{\mathcal{F}_{t+\alpha^*}}\left[p^*\right] + \int_{t+\alpha^*}^{t+\alpha^*+\delta^*} \eta_s^* dW_s.$$

We introduce the functions

$$\sigma(s) = \begin{cases} \sigma^*\left(s\right), & \text{if } s \in [t, t + \alpha^*], \\ t + \alpha^*, & \text{if } s \in (t + \alpha^*, t + \alpha^* + \delta^*], \end{cases} \tag{29}$$

$$u\left(s\right) = \begin{cases} u^*\left(s\right), & \text{if } s \in [t, t + \alpha^*], \\ \widetilde{u}\left(s\right), & \text{if } s \in (t + \alpha^*, t + \alpha^* + \delta^*], \end{cases}$$

$$\varphi\left(s\right) = \begin{cases} \varphi^*\left(s\right), & \text{if } s \in [t, t + \alpha^*], \\ \frac{1}{\sqrt{\delta^*}} E^{\mathcal{F}_{t+\alpha^*}}\left[p^*\right], & \text{if } s \in (t + \alpha^*, t + \alpha^* + \delta^*], \end{cases} \tag{30}$$

$$\psi\left(s\right) = \begin{cases} \psi^*\left(s\right), & \text{if } s \in [t, t + \alpha^*], \\ \sqrt{\delta^*} \eta_s^*, & \text{if } s \in (t + \alpha^*, t + \alpha^* + \delta^*], \end{cases}$$

$$\theta\left(s, r\right) = \begin{cases} \theta^*\left(s, r\right), & \text{if } t \leq r < s \leq t + \alpha^*, \\ \left(s - t - \alpha^*\right) + \theta^*\left(t + \alpha^*, r\right), & \\ & \text{if } t < r < t + \alpha^* < s \leq t + \alpha^* + \delta^*, \\ 0, & \text{if } t + \alpha^* \leq r < s \leq t + \alpha^* + \delta^* \end{cases}$$

and

$$Y(s) = \begin{cases} Y^*(s), \text{ if } s \in [t, t+\alpha^*], \\ S(s-t-\alpha^*)Y^*(t+\alpha^*) + \int_{t+\alpha^*}^s S(s-r) f(Y^*(t+\alpha^*), u(r)) dr \\ \quad + \int_{t+\alpha^*}^s S(s-r) g(Y^*(t+\alpha^*), u(r)) dW_r + \int_{t+\alpha^*}^s \varphi(r) dr \\ \quad + \int_{t+\alpha^*}^s \psi(r) dW_r, \text{ if } s \in (t+\alpha^*, t+\alpha^*+\delta^*]. \end{cases}$$

It suffices now to choose ε' as in Step 1 to prove that $(\sigma, u, \varphi, \psi, \theta, Y)$ is an element of $\mathcal{S}$. Moreover,

$$(\sigma^*, u^*, \varphi^*, \psi^*, \theta^*, Y^*) \preceq (\sigma, u, \varphi, \psi, \theta, Y) \text{ and}$$
$$\mathcal{N}\left((\sigma^*, u^*, \varphi^*, \psi^*, \theta^*, Y^*)\right) < \mathcal{N}\left((\sigma, u, \varphi, \psi, \theta, Y)\right).$$

This inequalities come in contradiction with the initial assumption of $(\sigma^*, u^*, \varphi^*, \psi^*, Y^*)$ being maximal. We deduce that $(\sigma^*, u^*, \varphi^*, \psi^*, Y^*)$ is an $\varepsilon-$approximate mild solution defined on $\left[t, \widetilde{T}\right]$ and this completes the proof of our Lemma.

Using this result, we are now able to prove that the quasi-tangency condition (2) written for $\lambda = 0$ provides a sufficient condition for ε-viability.

Proof (of Theorem 2). We assume that K satisfies the quasi-tangency condition (2) with $\lambda = 0$. Let us fix $t \in [0, T)$, $\xi \in L^2(\Omega, \mathcal{F}_t, P; K)$ and $\varepsilon \in (0, 1)$. We apply the previous Lemma and get the existence of an ε-approximate mild solution of (1) denoted $(\sigma, u, \varphi, \psi, \theta, Y)$ which is defined on $[t, T]$. From the definition of ε-approximate mild solutions,

$$Y(s) = S(s-t)\xi + \int_t^s S(s-r) f(Y(\sigma(r)), u(r)) dr$$

$$+ \int_t^s S(s-r) g(Y(\sigma(r)), u(r)) dW_r$$

$$+ \int_t^s S(\theta(s,r)) \varphi(r) dr + \int_t^s S(\theta(s,r)) \psi(r) dW_r, \qquad (31)$$

$dP - a.s.$, for all $s \in [t, T]$, $Y(\sigma(s)) \in K$, dP-almost surely, for all $s \in [t, T]$ and $Y(T) \in K$, dP-almost surely. Moreover,

$$E\left[|Y(\sigma(s)) - Y(s)|^2\right] \leq \varepsilon, \qquad (32)$$

for all $s \in [t, T]$. It follows that

$$d^2\left(X^{t,\xi,u}(s), L^2(\Omega, \mathcal{F}_s, P; K)\right) \leq E\left[\left|X^{t,\xi,u}(s) - Y(\sigma(s))\right|^2\right]$$

$$\leq 2\left(E\left[|Y(s) - Y(\sigma(s))|^2\right] + E\left[\left|X^{t,\xi,u}(s) - Y(s)\right|^2\right]\right), \qquad (33)$$

for all $s \in [t,T]$. Next, in order to estimate $E\left[\left|X^{t,\xi,u}(s) - Y(s)\right|^2\right]$, we use

$$
E\left[\left|X^{t,\xi,u}(s) - Y(s)\right|^2\right]
$$
$$
\leq C\left(E\left[\left|\int_t^s S(s-r)\left(f(Y(r),u(r)) - f\left(X^{t,\xi,u}(r),u(r)\right)\right)dr\right|^2\right]\right.
$$
$$
+ E\left[\left|\int_t^s S(s-r)\left(f(Y(r),u(r)) - f(Y(\sigma(r)),u(r))\right)dr\right|^2\right]
$$
$$
+ E\left[\left|\int_t^s S(s-r)\left(g(Y(r),u(r)) - g\left(X^{t,\xi,u}(r),u(r)\right)\right)dW_r\right|^2\right]
$$
$$
+ E\left[\left|\int_t^s S(s-r)\left(f(Y(r),u(r)) - f(Y(\sigma(r)),u(r))\right)dW_r\right|^2\right]
$$
$$
\left. + E\left[\left|\int_t^s S(\theta(s,r))\varphi(s)ds\right|^2\right] + E\left[\left|\int_t^s S(\theta(s,r))\psi(s)dW_s\right|^2\right]\right)
$$
$$
= C\left(I_1 + I_2 + I_3 + I_4 + I_5 + I_6\right), \tag{34}
$$

for all $s \in [t,T]$. In order to estimate I_1, we use the Cauchy-Schwarz inequality and the Lipschitz property of f, and get

$$
I_1 \leq CE\left[\left(\int_t^s \left|Y(r) - X^{t,\xi,u}(r)\right|dr\right)^2\right]
$$
$$
\leq C(s-t)^{1+2\gamma}\int_t^s (s-r)^{-2\gamma} E\left[\left|Y(r) - X^{t,\xi,u}(r)\right|^2\right]dr. \tag{35}
$$

For I_2, similar arguments combined with (32) yield

$$
I_2 \leq C\varepsilon. \tag{36}
$$

The assumption (A2) gives

$$
I_3 = E\left[\int_t^s \left|S(s-r)\left(g(Y(r),u(r)) - g\left(X^{t,\xi,u}(r),u(r)\right)\right)\right|^2 dr\right]
$$
$$
\leq C\int_t^s ((s-r)\wedge 1)^{-2\gamma} E\left[\left|Y(r) - X^{t,\xi,u}(r)\right|^2\right]dr \tag{37}
$$

and

$$
I_4 \leq C\varepsilon. \tag{38}
$$

For the last terms $I_{5,6}$ it suffices to recall properties (c) and (d) of Definition 3 and get

$$
I_5 + I_6 \leq C\varepsilon. \tag{39}
$$

We substitute the estimates (35)-(39) in (34) to obtain

$$E\left[\left|X^{t,\xi,u}\left(s\right)-Y\left(s\right)\right|^{2}\right]$$
$$\leq C\left(\varepsilon+\int_{t}^{s}\left(s-r\right)^{-2\gamma}E\left[\left|Y\left(r\right)-X^{t,\xi,u}\left(r\right)\right|^{2}\right]dr\right),$$

for all $s\in[t,T]$. Then, applying a variant of Gronwall's inequality, we have

$$E\left[\left|X^{t,\xi,u}\left(s\right)-Y\left(s\right)\right|^{2}\right]\leq C\varepsilon, \tag{40}$$

for all $s\in[t,T]$. We substitute (32) and (40) in (33) to finally get

$$d^{2}\left(X^{t,\xi,u}\left(s\right),L^{2}\left(\Omega,\mathcal{F}_{s},P;K\right)\right)\leq C\varepsilon,$$

for all $s\in[t,T]$. The conclusion follows by recalling that C can be chosen independent of ε (and $s\in[t,T]$) and $\varepsilon\in(0,1)$ is arbitrary. The proof of the main result is now complete.

4 Applications

4.1 The linear case

Let us now consider the following particular case of the control system (1):

$$\begin{cases} dX^{t,\xi,u}\left(s\right)=\left(AX^{t,\xi,u}\left(s\right)+Bu\left(s\right)\right)ds+\left(CX^{t,\xi,u}\left(s\right)+Du\left(s\right)\right)dW_{s}, \\ \text{for } s\in[t,T], \\ X^{t,\xi,u}\left(t\right)=\xi\in L^{2}\left(\Omega,\mathcal{F}_{t},P;H\right). \end{cases} \tag{41}$$

Here A is a linear unbounded operator on H that generates a C_{0}-semigroup of linear operators $\left(S\left(t\right)\right)_{t\geq0}$, $B\in\mathcal{L}\left(G;H\right)$, C is an $\mathcal{L}\left(\Xi;H\right)$-valued linear operator on H and there exist $\gamma\in\left[0,\frac{1}{2}\right)$ and $c>0$ such that, for every $t>0$, $S\left(t\right)C\in\mathcal{L}\left(H;L_{2}\left(\Xi;H\right)\right)$ and $\left|S\left(t\right)C\right|_{\mathcal{L}\left(H;L_{2}\left(\Xi;H\right)\right)}\leq c\left(t\wedge1\right)^{-\gamma}$, and D is an $L_{2}\left(\Xi;H\right)$-valued linear bounded operator on G. We also suppose that

$$U \text{ is a closed, bounded and convex subset of } G. \tag{A3}$$

Remark 4 If the assumption (A3) holds true, the space of admissible control processes $\mathcal{A}$ is convex. As a consequence, $\mathcal{A}$ is a closed subspace of $L^{2}\left(\left[t,T\right]\times\Omega;G\right)$ with respect to the weak topology on $L^{2}\left(\left[t,T\right]\times\Omega;G\right)$.

It is obvious that viability implies ε-viability for a closed set $K\subset H$. For the particular case of a linear control system we will prove that the quasi-tangency condition written for $\lambda=0$ is a sufficient condition not only for the $\varepsilon-$viability, but also for the viability property of an arbitrary nonempty, closed and convex set $K\subset H$. Hence, whenever $C\in\mathcal{L}\left(H;L_{2}\left(\Xi;H\right)\right)$, viability and ε-viability of closed, convex sets with respect to the linear control system (41) are equivalent. The main result of this section is

Theorem 4 *Let us suppose that (A1), (A2) and (A3) hold true. Moreover, we suppose that $K \subset H$ is a nonempty, closed and convex set that satisfies the quasi-tangency condition with respect to the control system (41) with $\lambda = 0$. Then K is viable with respect to the control system (41).*

Proof Let us fix $t \in [0, T)$ and $\xi \in L^2(\Omega, \mathcal{F}_t, P; K)$. For every $n \geq 2$, Lemma 1 gives the existence of an n^{-1}-approximate mild solution denoted by $(\sigma_n, u_n, \varphi_n, \psi_n, Y_n)$ and defined on $[t, T]$. By definition, for every $n \geq 2$ and every $s \in [t, T]$,

$$
Y_n(s) = S(s - t)\xi + \int_t^s S(s - r) B u_n(r) \, dr + \int_t^s S(s - r) C Y_n(\sigma_n(r)) \, dW_r
$$
$$
+ \int_t^s S(s - r) D u_n(r) \, dW_r
$$
$$
+ \int_t^s S(\theta_n(s, r)) \varphi_n(r) \, dr + \int_t^s S(\theta_n(s, r)) \psi_n(r) \, dW_r, \qquad (42)
$$

dP-almost surely. The estimates of Proposition 2 yield

$$
\sup_{n \geq 2} \sup_{s \in [t, T]} E\left[|Y_n(s)|^2 \right] \leq c,
$$

for a generic constant c. Moreover, since U is bounded,

$$
\sup_{n \geq 2} \sup_{s \in [t, T]} E\left[|u_n(s)|^2 \right] \leq c.
$$

The above estimates, together with the assumption (A3), allow to find a subsequence (still denoted by (Y_n, u_n)), a process $Y \in L^2\left([t, T]; L^2(\Omega; H)\right)$ and an admissible control process u such that $(Y_n, u_n) \to (Y, u)$ in the weak topology on $L^2\left([t, T]; L^2(\Omega; H \times U)\right)$.

   Step 1. We begin by showing that Y can be identified with $X^{t, \xi, u}$ (the unique mild solution of (41) starting from ξ and associated to the control process u). We make the following notations

$$
M_s^{1,n} = \int_t^s S(s - r) B u_n(r) \, dr, \ \ M_s^1 = \int_t^s S(s - r) B u(r) \, dr,
$$
$$
M_s^{2,n} = \int_t^s S(s - r) C Y_n(\sigma_n(r)) \, dW_r, \ \ M_s^2 = \int_t^s S(s - r) C Y(r) \, dW_r,
$$
$$
M_s^{3,n} = \int_t^s S(s - r) D u_n(r) \, dW_r, \ \ M_s^3 = \int_t^s S(s - r) D u(r) \, dW_r,
$$
$$
M_s^{4,n} = \int_t^s S(\theta_n(s, r)) \varphi_n(r) \, dr + \int_t^s S(\theta_n(s, r)) \psi_n(r) \, dW_r, \text{ for all } s \in [t, T].
$$

Let us fix $\phi \in L^2\left(\Omega, \mathcal{F}_s, P; H\right)$. Using the weak convergence of (u_n), one gets

$$\lim_{n \to \infty} E\left[\left\langle M_s^{1,n}, \phi\right\rangle\right] = \lim_{n \to \infty} E\left[\int_t^s \left\langle S\left(s-r\right) B u_n\left(r\right), \phi\right\rangle dr\right]$$

$$= \lim_{n \to \infty} E\left[\int_t^s \left\langle u_n\left(r\right), B^* S^*\left(s-r\right) \phi\right\rangle dr\right]$$

$$= E\left[\int_t^s \left\langle u\left(r\right), B^* S^*\left(s-r\right) \phi\right\rangle dr\right]$$

$$= E\left[\left\langle M_s^1, \phi\right\rangle\right].$$

If $\Phi \in L^2\left([t,T]; L^2\left(\Omega; H\right)\right)$ is an $(\mathcal{F}_t)$-adapted process, the previous equality, combined with a dominated convergence argument, allows to prove that

$$\lim_n E\left[\int_t^T \left\langle M_s^{1,n}, \Phi\left(s\right)\right\rangle ds\right] = E\left[\int_t^T \left\langle M_s^1, \Phi\left(s\right)\right\rangle ds\right].$$

Since Φ is arbitrary, this proves that $M^{1,n}$ converges in the weak topology on $L^2\left([t,T]; L^2\left(\Omega; H\right)\right)$ to M^1. Using condition (g) in the Definition 3 of approximate mild solutions, we get

$$\sup_{s \in [t,T]} E\left[\left|Y_n\left(s\right) - Y_n\left(\sigma_n\left(s\right)\right)\right|^2\right] \le n^{-1}.$$

Thus, in order to prove that $M^{2,n}$ converges in the weak topology on $L^2\left([t,T]; L^2\left(\Omega; H\right)\right)$ to M^2, one can replace $M^{2,n}$ by the process $N^{2,n}$ given by

$$N_s^{2,n} = \int_t^s S\left(s-r\right) C Y_n\left(r\right) dW_r, \text{ for all } s \in [t,T].$$

If $\phi \in L^2\left(\Omega, \mathcal{F}_s, P; H\right)$, using the martingale representation theorem we prove

$$\lim_{n \to \infty} E\left[\left\langle N_s^{2,n}, \phi\right\rangle\right] = E\left[\left\langle M_s^2, \phi\right\rangle\right].$$

Using, as before, the dominated convergence theorem, we get that $N^{2,n}$ converges in the weak topology on $L^2\left([t,T]; L^2\left(\Omega; H\right)\right)$ to M^2. Similar arguments allow to prove that $M^{3,n}$ converges in the weak topology on $L^2\left([t,T]; L^2\left(\Omega; H\right)\right)$ to M^3. The conditions (c) and (d) in the Definition 3 of approximate mild solutions imply that $M^{4,n}$ converges strongly to 0 in $L^2\left([t,T]; L^2\left(\Omega; H\right)\right)$. It follows that the limit (Y, u) satisfies

$$Y\left(s\right) = S\left(s-t\right) \xi + \int_t^s S\left(s-r\right) B u\left(r\right) dr + \int_t^s S\left(s-r\right) C Y\left(r\right) dW_r$$

$$+ \int_t^s S\left(s-r\right) D u\left(r\right) dW_r. \tag{43}$$

This equation is, a priori, satisfied $dPds-$almost everywhere. We can now identify Y with its continuous version $X^{t,\xi,u}$.

$\underline{\text{Step 2.}}$ We claim that $Y_s \in K \ dPds-$ almost everywhere on $\Omega \times [t,T]$. Indeed, let us consider the following application $\gamma : L^2 \left([t,T]; L^2 (\Omega; H)\right) \longrightarrow \mathbb{R}_+$,

$$\gamma (Z) = E \left[\int_t^T d_K^2 \left(Z \left(s \right) \right) ds \right].$$

Obviously, this application is convex. Using the fact that Y_n converges in the weak topology on $L^2 \left([t,T]; L^2 (\Omega; H)\right)$ to Y, one finds a sequence of convex combinations of (Y_n), denoted by Z_n, which converges strongly to Y in $L^2 \left([t,T]; L^2 (\Omega; H)\right)$. For every $n \geq 2$,

$$\gamma (Y_n) \leq E \left[\int_t^T \left| Y_n \left(s \right) - Y_n \left(\sigma \left(s \right) \right) \right|^2 ds \right] \leq T n^{-1}.$$

Thus, using the convexity of γ, one can assume, without loss of generality, that

$$\gamma (Z_n) \leq n^{-1}, \tag{44}$$

for all $n \geq 2$. Then

$$\gamma (Y) \leq 2 \left(\gamma (Z_n) + E \left[\int_t^T \left| Y \left(s \right) - Z_n \left(s \right) \right|^2 ds \right] \right).$$

We let $n \rightarrow \infty$ in the last inequality. Due to (44) and to the strong convergence of Z_n to Y, one has

$$\gamma (Y) = 0.$$

In other words, $Y (s) \in K$, $dPds-$ almost everywhere on $\Omega \times [t,T]$. The conclusion follows from the continuity of Y.

4.2 Nagumo's stochastic theorem. Viability of smooth sets

We assume that A is a linear operator on H that generates a semigroup of continuous operators denoted by $(S(t))_{t \geq 0}$. We consider $F : H \longrightarrow H$ and $G : H \longrightarrow L_2 (\Xi; H)$ such that, for some positive constant $c > 0$,

$$\left| F \left(x \right) - F \left(y \right) \right| + \left| G(x) - G(y) \right|_{L_2(\Xi;H)} \leq c \left| x - y \right|. \tag{B}$$

for all $x, y \in H$. We consider the stochastic semilinear equation

$$\begin{cases} dX^{t,\xi}(s) = \left(AX^{t,\xi}(s) + F \left(X^{t,\xi}(s) \right) \right) ds \\ \quad + G \left(X^{t,\xi}(s) \right) dW_s, \text{ for all } s \in [t,T], \\ \quad X^{t,\xi}(t) = \xi \in L^2 \left(\Omega, \mathcal{F}_t, P; H \right). \end{cases} \tag{45}$$

The aim of this subsection is to deduce explicit conditions for the viability of smooth sets with respect to the system (45). In particular, we study the viability of the closed unit ball $\overline{B}(0,1)$.

We begin by stating the following version of Nagumo's stochastic theorem.

Theorem 5 *(Nagumo's stochastic theorem) A nonempty, closed set $K \subset H$ is viable with respect to (45) if and only if, for every $t \in [0, T)$ and every $\xi \in L^2(\Omega, \mathcal{F}_t, P; K)$, there exist a sequence $h_n \searrow 0$ and a sequence of random variables $p_n \in L^2(\Omega, \mathcal{F}_{t+h_n}, P; H)$ such that*

$$\lim_n E\left[|p_n|^2\right] + \frac{1}{h_n} E\left[\left|E^{\mathcal{F}_t}[p_n]\right|^2\right] = 0$$

and $S(h_n)\xi + h_n F(\xi) + G(\xi)(W_{t+h_n} - W_t) + \sqrt{h_n} p_n \in K$, $dP-almost surely, for all n.

This result is a simple consequence of Theorems 1 and 2.

We consider some orthonormal basis in Ξ, denoted by $(e_m)_{m \in \mathbb{N}^*}$. We introduce, for every $m \in \mathbb{N}^*$, $\beta^m_\cdot = \langle W_\cdot, e_m \rangle$. This defines a sequence of independent standard 1-dimensional Brownian motions. We also consider, for every $m \in \mathbb{N}^*$, the $m-$dimensional Brownian motion $W^m_\cdot = \sum_{i=1}^m \beta^i_\cdot e_i$. For every $m \geq 1$, we denote by $\{\mathcal{F}^m_t : t \geq 0\}$ the filtration generated by the one-dimensional Brownian motion $\beta^m_\cdot$ completed by the P-null sets and by $\left\{\mathcal{F}^{1,m}_t : t \geq 0\right\}$ the filtration generated by the finite-dimensional Brownian motion $W^m_\cdot$ completed by the P-null sets.

We will show how deterministic criteria can be inferred from the quasi-tangency condition for smooth sets of constraints. To be more precise, we consider sets of constraints K that can be written as

$$K = \{x \in H : \varphi(x) \leq 0\}, \tag{46}$$

for some C^2-function $\varphi : H \longrightarrow \mathbb{R}$ with bounded, Lipschitz-continuous second order derivative.

Proposition 4 *Under the assumption (B), if the set K given by (46) is viable with respect to (45), then, for every $x \in D(A)$ such that $\varphi(x) = 0$,*

$$0 \geq \langle D\varphi(x), Ax \rangle + \langle D\varphi(x), F(x) \rangle$$

$$+ \frac{1}{2} \sum_{i=1}^{\infty} \langle D^2\varphi(x) G(x) e_i, G(x) e_i \rangle, \tag{47}$$

and

$$G^*(x) D\varphi(x) = 0. \tag{48}$$

Proof If the set K is viable, and $x \in D(A)$ such that $\varphi(x) = 0$, for every sequence $h_n \searrow 0$ we get, by Nagumo's stochastic theorem (see also Remark 3), the existence of some sequence $p_n \in L^2(\Omega, \mathcal{F}_{h_n}, P; H)$ such that

$$\begin{cases} \lim_n \left(E\left[|p_n|^2\right] + \frac{1}{h_n} E\left[\left|E^{\mathcal{F}_t}[p_n]\right|^2\right] \right) = 0 \text{ and} \\ S(h_n) x + h_n F(x) + G(x) W_{h_n} + \sqrt{h_n} p_n \in K, \ dP\text{-a.s.}, \end{cases}$$

for every $n \geq 1$. Moreover, we can assume (cf. Remark 3) that $p_n \in L^4(\Omega, \mathcal{F}_{h_n}, P; H)$, for every $n \geq 1$. Using the martingale representation

theorem, there exists an $(\mathcal{F})$-predictable process q_n such that $p_n = E[p_n] + \int_0^{h_n} q_n(s) \, dW_s$. Moreover, by standard estimates,

$$E\left[\left(\int_0^{h_n} |q_n(s)|^2 \, ds\right)^2\right] \leq CE\left[|p_n|^4\right], \qquad (49)$$

for some generic constant C independent of p_n and h_n. For every $n \in \mathbb{N}^*$, we introduce the process $Y^{x,n}_{\cdot}$ given by

$$Y^{x,n}_s = S(h_n)x + h_n F(x) + G(x) W_s + \sqrt{h_n} E[p_n] + \sqrt{h_n} \int_0^s q_n(r) \, dW_r,$$

for every $s \in [0, h_n]$. It is obvious from (49) that

$$E\left[\sup_{s \in [0, h_n]} |Y^{x,n}_s|^4\right] \leq C.$$

We apply Itô's formula to $Y^{x,n}$ to get

$$\varphi\left(Y^{x,n}_{h_n}\right)$$
$$= \varphi\left(S(h_n)x + h_n F(x) + \sqrt{h_n} E[p_n]\right)$$
$$+ \int_0^{h_n} \left\langle D\varphi(Y^{x,n}_s), \left(G(x) + \sqrt{h_n} q_n(s)\right) dW_s \right\rangle$$
$$+ \frac{1}{2} \int_0^{h_n} \sum_{i=1}^{\infty} \left\langle D^2\varphi(Y^{x,n}_s)\left(G(x) + \sqrt{h_n} q_n(s)\right) e_i, \left(G(x) + \sqrt{h_n} q_n(s)\right) e_i \right\rangle ds.$$
$$(50)$$

Using a first order Taylor expansion of φ gives

$$\varphi\left(S(h_n)x + h_n F(x) + \sqrt{h_n} E[p_n]\right)$$
$$\geq \langle D\varphi(x), (S(h_n) - I)x + h_n F(x)\rangle - h_n P^1_n, \qquad (51)$$

where $P^1_n \in \mathbb{R}_+$ such that $\lim_n P^1_n = 0$. Moreover, by standard estimates,

$$\int_0^{h_n} \sum_{i=1}^{\infty} \left\langle D^2\varphi(Y^{x,n}_s)\left(G(x) + \sqrt{h_n} q_n(s)\right) e_i, \left(G(x) + \sqrt{h_n} q_n(s)\right) e_i \right\rangle ds$$
$$\geq h_n \sum_{i=1}^{\infty} \left\langle D^2\varphi(x) G(x) e_i, G(x) e_i \right\rangle - h_n P^2_n, \qquad (52)$$

where $P^2_n \in L^1(\Omega, \mathcal{F}_{h_n}, P; \mathbb{R})$ such that $\lim_n E\left[|P^2_n|\right] = 0$. Substituting (51) and (52) in (50) and taking expectation, we get

$$0 \geq E\left[\varphi\left(Y^{x,n}_{h_n}\right)\right] \geq \langle D\varphi(x), (S(h_n) - I)x + h_n F(x)\rangle$$
$$+ h_n \sum_{i=1}^{\infty} \left\langle D^2\varphi(x) G(x) e_i, G(x) e_i \right\rangle - h_n \left(P^1_n + P^2_n\right).$$

The condition (47) follows by dividing the last inequality by h_n and letting $n \to \infty$.

Standard estimates and (50) yield

$$0 \geq \varphi \left(Y_{h_n}^{x,n} \right) = \int_0^{h_n} \left\langle D\varphi \left(Y_s^{x,n} \right), \left(G(x) + \sqrt{h_n} q_n(s) \right) dW_s \right\rangle$$
$$- \sqrt{h_n} Q_n^1, \tag{53}$$

where $Q_n^1 \in L^1 \left(\Omega, \mathcal{F}_{h_n}, P; \mathbb{R} \right)$ such that $lim_n E \left[\left| Q_n^1 \right| \right] = 0$. Moreover, again by standard estimates,

$$\int_0^{h_n} \left\langle D\varphi \left(Y_s^{x,n} \right), \left(G(x) + \sqrt{h_n} q_n(s) \right) dW_s \right\rangle$$
$$= \left\langle D\varphi(x), G(x) W_{h_n} \right\rangle - \sqrt{h_n} Q_n^2, \tag{54}$$

where $Q_n^2 \in L^1 \left(\Omega, \mathcal{F}_{h_n}, P; H \right)$ such that $lim_n E \left[\left| Q_n^2 \right| \right] = 0$. Then, by substituting (54) in (53) and dividing by $\sqrt{h_n}$ we get

$$Q_n^1 + Q_n^2 \geq \frac{1}{\sqrt{h_n}} \left\langle D\varphi(x), G(x) W_{h_n} \right\rangle .$$

For every $m \in \mathbb{N}^*$, taking conditional expectation with respect to $\mathcal{F}_\infty^m$ yields

$$Q_n \geq \left\langle D\varphi(x), G(x) e_m \right\rangle \frac{\beta_{h_n}^m}{\sqrt{h_n}},$$

for some process $Q_n \in L^1 \left(\Omega, \mathcal{F}_{h_n}, P; H \right)$ such that $lim_n E \left[|Q_n| \right] = 0$. Standard properties of Brownian sample paths imply that the above inequality can only hold if (48) holds true. The proof is now complete.

Using the above result, we are going to deduce explicit deterministic conditions for the viability of the unit ball. In the case of a diagonal operator A, using the quasi-tangency characterization of viability, we prove that these conditions are also sufficient.

Proposition 5 *If the unit ball $\overline{B}(0,1)$ is viable with respect to (45), then, for every $x \in D(A)$ such that $|x| = 1$,*

$$\begin{cases} 0 \geq \langle x, Ax \rangle + \langle x, F(x) \rangle + \frac{1}{2} |G(x)|_{L_2(\Xi;H)}^2, & and \\ G^*(x) x = 0. \end{cases} \tag{55}$$

Conversely, if A is a diagonal operator, this condition is also sufficient for the viability of $\overline{B}(0,1)$.

Proof That viability implies (55) is a simple consequence of Proposition 4. Conversely, we assume that (55) holds true whenever $x \in D(A)$ is such that $|x| = 1$. We fix $t \in [0,T)$ and $\xi \in L^2 \left(\Omega, \mathcal{F}_t, P; \overline{B}(0,1) \right)$. For every $l \geq 1$, we denote by J_l the orthogonal projector onto the space H_l generated by the

first l eigenvectors of A. We fix, for the moment $m, l \geq 1$. For every $t \geq 0$ and every $\zeta \in L^2 \left(\Omega, \mathcal{F}_t^{1,m}, P; H_l \right)$, we define the system

$$
\begin{cases}
X_s^{t,\zeta,m,l} = \left(A_l X_s^{t,\zeta,m,l} + J_l F \left(X_s^{t,\zeta,m,l} \right) \right) ds + J_l G \left(X_s^{t,\zeta,m,l} \right) dW_s^m, \ s \in [t,T], \\
X_t^{t,\xi,m,l} = \zeta.
\end{cases}
$$

Obviously, the solution lives in the finite dimensional space H_l and the Brownian motion $W_.^m$ is finite dimensional. Consequently, the condition (55) and classical results due to Aubin, Da Prato in the finite dimensional framework (cf. [2], page 596) yield that the unit ball of H_l, denoted by $\overline{B}_{H_l}(0,1)$ is viable with respect to the previous equation. Then, from Theorem 1, for every $h > 0$, there exists $p_{l,m,h} \in L^2 \left(\Omega, \mathcal{F}_{t+h}^{1,m}, P; H_l \right) \subset L^2 \left(\Omega, \mathcal{F}_{t+h}, P; H \right)$ such that

$$
S(h) J_l E^{\mathcal{F}_\infty^{1,m}} [\xi] + \int_t^{t+h} S(t+h-s) J_l F \left(J_l E^{\mathcal{F}_\infty^{1,m}} [\xi] \right) ds
$$

$$
+ \int_t^{t+h} S(t+h-s) J_l G \left(J_l E^{\mathcal{F}_\infty^{1,m}} [\xi] \right) dW_s^m + \sqrt{h} p_{l,m,h} \in \overline{B}(0,1), \quad (56)
$$

$dP - a.s.$ Moreover (see inequality (11)),

$$
E \left[|p_{l,m,h}|^2 \right] + \frac{1}{h} E \left[\left| E^{\mathcal{F}_t} [p_{l,m,h}] \right|^2 \right]
$$

$$
\leq C \left(\sup_{s \in [0,h]} E \left[\left| (S(s) - I) J_l E^{\mathcal{F}_\infty^{1,m}} [\xi] \right|^2 \right] + h \right)
$$

$$
\leq C \left(\sup_{s \in [0,h]} E \left[|(S(s) - I) \xi|^2 \right] + h \right). \quad (57)
$$

A careful look at the proof of Theorem 1 shows that C can be chosen independent of l and m. The inequality (57) guarantees the existence of some subsequence $(p_{l,m,h})_{l,m}$ converging weakly in $L^2 (\Omega, \mathcal{F}_{t+h}, P; H)$ to some $p_h \in L^2 (\Omega, \mathcal{F}_{t+h}, P; H)$ such that

$$
E \left[|p_h|^2 \right] + \frac{1}{h} E \left[\left| E^{\mathcal{F}_t} [p_h] \right|^2 \right] \leq C \left(E \left[\sup_{s \in [0,h]} |S(s) \xi - \xi|^2 \right] + h \right). \quad (58)
$$

Then, by taking weak limit in $L^2 (\Omega, \mathcal{F}_{t+h}, P; H)$ in (56), one gets

$$
S(h) \xi + \int_t^{t+h} S(t+h-s) F(\xi) ds
$$

$$
+ \int_t^{t+h} S(t+h-s) G(\xi) dW_s + \sqrt{h} p_h \in \overline{B}(0,1). \quad (59)
$$

We recall that t, ξ, h are arbitrary and the constant C in (58) can be chosen independent of h. Then (59) and (58) give the 0 quasi-tangency condition. The viability of the initial system follows from Theorem 2.

References

1. J.,-P., Aubin, *Viability Theory*, Birkhäuser (1992).
2. J.,-P., Aubin, G., Da Prato, Stochastic viability and invariance, Ann. Scuola Norm. Pisa 27 (1990), pp. 595-694.
3. J.,-P., Aubin, H., Frankowska, *Set-valued analysis, Systems& Control: Foundations& Applications*, vol. 2. Birkhäuser Boston Inc., Boston, MA (1990).
4. R., Buckdahn, P., Cardaliaguet, M., Quincampoix, *A representation formula for the mean-curvature motion*, SIAM J. Math. Anal. 33 (2001), pp. 827-846 .
5. R., Buckdahn, S., Peng, M., Quincampoix, C., Rainer, *Existence of stochastic control under state constraints*, C.R. Acad. Sci. Paris Sr. I Math. 327 (1998), pp. 17-22.
6. R., Buckdahn, M., Quincampoix, A., Rascanu, *Viability property for backward stochastic differential equation and applications to partial differential equations,* Probab. Theory Related Fields 116 (2000), pp. 485-504.
7. R., Buckdahn, M., Quincampoix, G., Tessitore, *Controlled stochastic differential equations under constraints in infinite dimensional spaces.* SIAM J. Control Optim. 47 (2008), no. 1, pp. 218–250.
8. O., Carja, I.I., Vrabie, *Some new viability results for semilinear differential inclusions*, NoDEA Nonlinear Differential Equations Appl., 4 (1997), pp. 401-424.
9. O., Carja, M., Necula, I.I., Vrabie, *Necessary and sufficient conditions for viability for semilinear differential inclusions,* Trans. Amer. Math. Soc. 361 (2009), pp. 343-390.
10. O., Carja, M., Necula, I.I., Vrabie, *Viability, Invariance and Applications,* North-Holland Mathematics Studies (2007).
11. G., Da Prato, J., Zabczyk, *Stochastic equations in infinite dimensions*, Cambridge University Press, Cambridge (1992).
12. M., Fuhrman, G., Tessitore, Nonlinear Kolmogorov equations in infinite dimensional spaces: the backward stochastic differential equations approach and applications to optimal control, Ann. Probab., 30 (2002), pp. 1397-1465.
13. S., Gautier, L., Thibault, *Viability for constrained stochastic differential equations,* Differential Integral Equations 6 (1993), pp. 433-466.
14. D., Goreac, *Non-compact-valued stochastic control under state constraints*, Bull. Sci. Math., vol. 131, no8 (2007), pp. 716-737.
15. D., Goreac, *Approximate Controllability for Linear Stochastic Differential Equations in Infinite Dimensions,* Appl. Math. Opt. (2009), DOI 10.1007/s00245-009-9068-y
16. M., Nagumo, *Uber die lage derintegralkurven gewhnlicher differentialgleichungen*, Proc. Phys. Math. Soc. Japan, 24 (1942), pp. 551-559.